JOURNAL OF APPROXIMATION THEORY 53, 295–303 (1988)

A Note on a Bernstein-Type Operator of Bleimann, Butzer, and Hahn

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Communicated by P. L. Butzer

Received August 24, 1985; revised October 24, 1986

Let $L_n(f, x)(f \in C[0, \infty))$ be a Bernstein-type approximation operator as defined and studied by Bleimann, Butzer, and Hahn. Probabilistic arguments are used to simplify and sharpen some of their results. The rates of convergence are given in terms of the first and second moduli of continuity. Moreover, an appropriate limit of L_n is shown to be the well-known Szasz operator. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let $C[0, \infty)$ be the space of continuous functions on the unbounded interval $[0, \infty)$, and let $f \in C[0, \infty)$. Bleimann, Butzer, and Hahn [1] introduced a Bernstein-type approximation operator defined by

$$L_n(f, x) = (1+x)^{-n} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k} x^k, \qquad n \in \mathbb{N}. \tag{1}$$

They proved that $L_n(f, x) \to f(x)$ (as $n \to \infty$) for each $x \in [0, \infty)$, and found a rate of convergence by estimating $|L_n(f, x) - f(x)|$ in terms of the second modulus of continuity of $f \in C_B[0, \infty)$. Here $C_B[0, \infty)$ is the space of bounded and uniformly continuous functions on $[0, \infty)$. The object of this note is to exploit probabilistic arguments to simplify and sharpen some of their results. The rates of convergence are given in terms of the first and second moduli of continuity. We show that $|L_n(f, x) - f(x)| \leqslant 3\omega(f, \sqrt{x(1+x)^2/n})$, where $\omega(f, \delta)$ is the first modulus of continuity of f. An improved version of Theorem 2 of [1] is given by showing that

$$|L_n(f, x) - f(x)| \leqslant 2C\left[\omega_2\left(f, \sqrt{\frac{x(1+x)^2}{n}}\right) + \frac{x(1+x)^2}{n}\|f\|\right],$$

0021-9045/88 $3.00

where $\omega_2(f, \delta)$ is the second modulus of continuity of $f \in C_B[0, \infty)$, and $\|f\| = \sup_{x \in [0, \infty)} |f(x)|$. Moreover, it is shown that an appropriate limit of L_n is the famous Szasz operator.

2. The Rates of Convergence

Let $y_1, y_2, \ldots$ be independent and identically distributed random variables such that $P(y_1 = 1) = p$, $P(y_1 = 0) = q$, where $p = x/(1+x)$ and $q = 1/(1+x)$, $x \in [0, \infty)$. To avoid trivialities let $x > 0$. Clearly, $S_n = y_1 + \cdots + y_n$ follows a binomial distribution $b(n, p)$ with parameters n and p, and

$$P(S_n = k) = \binom{n}{k} p^k q^{n-k}, \qquad k = 0, 1, 2, \ldots, n. \tag{2}$$

Set $X_n = S_n/(n - S_n + 1)$, $n = 1, 2, \ldots$, and note from (1) that $L_n(f, x) = Ef(X_n)$, where E denotes the expectation operator. Since $X_n \to p/q = x$ in probability as $n \to \infty$, then $L_n(f, x) \to f(x)$ (as $n \to \infty$) by the law of large numbers if $f \in C[0, \infty)$ (cf. Khan [5]). To obtain sharper results we will first compute EX_n and EX_n^2 and estimate $e_n(x) = E(X_n - x)^2$. It is rather trivial to show that

$$EX_n = x - xp^n \to x \qquad \text{as} \quad n \to \infty. \tag{3}$$

From (2) and the definition of X_n it follows that

$$\begin{aligned}
EX_n^2 &= \sum_{k=0}^{n} \frac{k^2}{(n-k+1)^2} P(S_n = k) \\
&= \sum_{k=1}^{n} \frac{k}{(n-k+1)(k-1)!} \frac{n!}{(n-k+1)!} p^k q^{n-k} \\
&= \sum_{j=0}^{n-1} \frac{(j+1)}{(n-j)j!} \frac{n!}{(n-j)!} p^{j+1} q^{n-j-1} \\
&= \sum_{j=1}^{n-1} \frac{n!\, p^{j+1} q^{n-j-1}}{(n-j)(j-1)!(n-j)!} + \sum_{j=0}^{n-1} \frac{n!\, p^{j+1} q^{n-j-1}}{(n-j)j!(n-j)!} \\
&= \frac{pq^{n-1}}{n} + \sum_{j=1}^{n-1} \frac{n! p^{j+1} q^{n-j-1}}{(n-j)(j-1)!(n-j)!} + \sum_{j=1}^{n-1} \frac{n! p^{j+1} q^{n-j-1}}{(n-j)j!(n-j)!}.
\end{aligned}$$

Letting $k=j-1$ and recalling that $p=1-q=x/(1+x)$ one finds that

$$EX_n^2=\frac{x}{n(1+x)^n}+\sum_{k=0}^{n-2}\binom{n}{k}p^{k+2}q^{n-k-2}\left[\frac{n-k}{n-k-1}+\frac{n-k}{(k+1)(n-k-1)}\right]$$

$$=\frac{x}{n(1+x)^n}+\sum_{k=0}^{n-2}\binom{n}{k}p^{k+2}q^{n-k-2}\frac{(n-k)(k+2)}{(n-k-1)(k+1)}.$$

Since

$$\frac{(n-k)(k+2)}{(n-k-1)(k+1)}=1+\frac{(n+1)}{(k+1)(n-k-1)},$$

then

$$EX_n^2=\frac{x}{n(1+x)^n}+\sum_{k=0}^{n-2}\binom{n}{k}p^{k+2}q^{n-k-2}+\sum_{k=0}^{n-2}\frac{(n+1)!\,p^{k+2}q^{n-k-2}}{(k+1)!\,(n-k)!(n-k-1)}$$

$$=\frac{x}{n(1+x)^n}+\sum_{k=0}^{n}\binom{n}{k}p^{k+2}q^{n-k-2}-p^{n+2}q^{-2}-np^{n+1}q^{-1}$$

$$+\sum_{k=0}^{n-2}\frac{(n+1)!\,p^{k+2}q^{n-k-2}}{(k+1)!(n-k)!(n-k-1)}$$

$$=\frac{x}{n(1+x)^n}+x^2-x^2\left(\frac{x}{1+x}\right)^n-nx\left(\frac{x}{1+x}\right)^n+R, \tag{4}$$

where

$$R=\sum_{k=0}^{n-2}\frac{(n+1)!\,p^{k+2}q^{n-k-2}}{(k+1)!\,(n-k)!(n-k-1)}.$$

Now we will analyze the term R. Setting $k+1=j$ we have

$$R=\sum_{j=1}^{n-1}\frac{(n+1)!\,p^{j+1}q^{n-j-1}}{j!\,(n-j+1)!\,(n-j)}=\sum_{j=1}^{n-1}\binom{n+1}{j}\frac{p^{j+1}q^{n-j-1}}{(n-j)}.$$

Since $(n-j)^{-1}=(n-j+2)^{-1}(1+2/(n-j))\leqslant 3/(n-j+2)$, $1\leqslant j\leqslant n-1$ $(n\geqslant 2)$, we have

$$R\leqslant 3\sum_{j=1}^{n-1}\binom{n+1}{j}\frac{p^{j+1}q^{n-j-1}}{(n-j+2)}\leqslant\frac{3p}{q^2}\sum_{j=0}^{n+1}\binom{n+1}{j}\frac{p^j q^{n+1-j}}{(n-j+2)}$$

$$\leqslant 3x(1+x)\,E((n+1-Y)+1)^{-1}=3x(1+x)\,E(\xi+1)^{-1},$$

where Y has binomial distribution $b(n+1, p)$, and $\xi = n+1-Y$ is $b(n+1, q)$, $q = 1-p$. Hence a result of Chao and Strawderman [2, p. 430] gives

$$R \leqslant 3x(1+x)(1-p^{n+2})/(n+2)\, q \leqslant 3x(1+x)^2/(n+2). \tag{5}$$

Letting $e_n(x) = E(X_n - x)^2$ it follows from (3), (4), and (5) that

$$\begin{aligned} e_n(x) &= \frac{x}{n(1+x)^n} + x^2\left(\frac{x}{1+x}\right)^n - nx\left(\frac{x}{1+x}\right)^n + R \\ &\leqslant \frac{x}{n(1+x)^n} + x^2\left(\frac{x}{1+x}\right)^n - nx\left(\frac{x}{1+x}\right)^n + \frac{3x(1+x)^2}{n+2}. \end{aligned}$$

Since $(x/(1+x))^n \leqslant (1+x)/n$, we have

$$e_n(x) \leqslant \frac{x}{n} + \frac{x^2(1+x)}{n} + \frac{3x(1+x)^2}{n} \leqslant \frac{4x(1+x)^2}{n}. \tag{6}$$

Let $C_B^*[0, \infty)$ be the space of continuous bounded functions on $[0, \infty)$. Obviously $C_B^*[0, \infty)$ is a larger space than $C_B[0, \infty)$ of Section 1. In order to establish our first result let $f \in C_B^*[0, \infty)$ and set $\omega(f, \delta) = \sup\{|f(x) - f(y)|: |x-y| \leqslant \delta,\ x, y \in [0, \infty)\}$, $\delta > 0$. The following theorem gives the rate of convergence in terms of the first modulus of continuity $\omega(f, \delta)$.

THEOREM 1. *Let* $L_n(f, x)$ *be defined by* (1) *and* $f \in C_B^*[0, \infty)$. *Then*

$$|L_n(f, x) - f(x)| \leqslant 3\omega(f, \sqrt{x(1+x)^2/n}), \qquad n \geqslant 1. \tag{7}$$

Proof. Let $\delta > 0$ and $\lambda = [|X_n - x|/\delta]$, $[a]$ denotes the greatest integer $\leqslant a$. Clearly, $|f(X_n) - f(x)| \leqslant (1+\lambda)\, \omega(f, \delta)$, and

$$|L_n(f, x) - f(x)| = |Ef(X_n) - f(x)| \leqslant \omega(f, \delta)(1 + E\lambda).$$

The inequality $E\lambda \leqslant \sqrt{E\lambda^2} \leqslant \sqrt{e_n(x)/\delta^2}$ combined with (6) gives

$$|L_n(f, x) - f(x)| \leqslant \omega(f, \delta)\left(1 + \frac{2(1+x)\sqrt{x}}{\delta\sqrt{n}}\right),$$

and (7) follows on taking $\delta = (1+x)\sqrt{x/n}$.

Theorem 1 can be strengthened by using the second modulus of continuity (cf. [1]). Let $\|\phi\| = \sup_{x \in [0, \infty)} |\phi(x)|$, where $\phi \in C_B[0, \infty)$. With

$\Delta^2 f = f(x+2t) - 2f(x+t) + f(x)$ ($f \in C_B[0, \infty)$) define the second modulus of continuity by

$$\omega_2(f, \delta) = \sup_{t:\, |t| \leqslant \delta} \|\Delta^2 f\|, \qquad \delta > 0.$$

Letting $h = X_n - x$ we obtain from (3) that

$$|Eh| \leqslant x(x/(1+x))^n \leqslant x(1+x)/n. \tag{8}$$

An improved version of a result due to Bleimann, Butzer, and Hahn is given below by dropping the unpleasant condition $n \geqslant N(x) = 24(1+x)$ in Theorem 2 of [1]. To this end we need the following elementary result.

Let $g \in C_B[0, \infty)$ be such that g' and $g'' \in C_B[0, \infty)$. With $h = X_n - x$ we note that

$$g(X_n) - g(x) = \int_0^h g'(x+t)\, dt = hg'(x+h) - \int_0^h tg''(x+t)\, dt.$$

Taking expectation and using (6) and (8) it is easy to see that

$$\begin{aligned} |L_n(g, x) - g(x)| &\leqslant |Eh|\ \|g'\| + \frac{1}{2} Eh^2\ \|g''\| \\ &\leqslant \frac{x(1+x)}{n} \|g'\| + \frac{2x(1+x)^2}{n} \|g''\|. \end{aligned}$$

Hence we have

$$|L_n(g, x) - g(x)| \leqslant \frac{2x(1+x)^2}{n} (\|g'\| + \|g''\|). \tag{9}$$

Using (9) in the proof of Theorem 2 of [1] we obtain the following stronger version.

THEOREM 2. *Let $f \in C_B[0, \infty)$, $x \in [0, \infty)$. Then for $n = 1, 2, ...$,*

$$|L_n(f, x) - f(x)| \leqslant 2C \left[\omega_2\left(f, \sqrt{\frac{x(1+x)^2}{n}}\right) + \frac{x(1+x)^2}{n} \|f\| \right],$$

where C is a constant.

Remark. Totik [6] proves a saturation class theorem giving a necessary and sufficient condition for $\sup_{x \geqslant 0} |L_n(f, x) - f(x)| = O(n^{-1})$. However, this saturation property (dependent on f' and f'') is, by no means, an improvement of Theorem 2.

3. A Limiting Property of L_n

Let $f \in C_B[0, \infty)$. Define the Szasz operator by

$$S_m(f, x) = e^{-mx} \sum_{k=0}^{\infty} f\left(\frac{k}{m}\right) \frac{(mx)^k}{k!}, \qquad x \geq 0, \tag{10}$$

where m is a fixed positive integer. That $S_m(f, x)$ is an appropriate limit of L_n is an interesting consequence. The following lemma is needed to prove this limiting property.

Lemma. *Let* $p_k(mn, x) = \binom{mn}{k}(x/n)^k(1 + x/n)^{-mn}$, $k = 0, 1, ..., mn$, *and* $\alpha_k(mx) = \exp(-mx)(mx)^k/k!$, $k = 0, 1, 2, ...$ *Then*

(i) $\alpha_k(mx) \exp(-k(k-1)/(mn-k+1)) \leq p_k(mn, x) \leq \alpha_k(mx) \exp(mx^2/(n+x))$,

(ii) $\sum_{k=0}^{mn} |p_k(mn, x) - \alpha_k(mx)| \to 0$ *as* $n \to \infty$,

(iii) $\max_{0 \leq k \leq mn} |p_k(mn, x) - \alpha_k(mx)| \to 0$ *as* $n \to \infty$.

Proof. Since $e^{-t} \geq 1 - t$ $(0 \leq t \leq 1)$, it follows that

$$p_k(mn, x) = \binom{mn}{k}\left(\frac{x}{n+x}\right)^k\left(1 - \frac{x}{n+x}\right)^{mn-k} \leq \frac{(mx)^k}{k!}\left(1 - \frac{x}{n+x}\right)^{mn}$$
$$\leq \alpha_k(mx) \exp(mx^2/(n+x)).$$

Since $(1 - \beta) \geq \exp(-\beta/(1-\beta))$ $(0 \leq \beta < 1)$, it follows that

$$p_k(mn, x) = \frac{(mx)^k}{k!} \prod_{i=1}^{k}\left(1 - \frac{i-1}{mn}\right)\left(1 - \frac{x}{n+x}\right)^{mn}$$
$$\geq \frac{(mx)^k}{k!}\left(1 - \frac{k-1}{mn}\right)^k\left(1 - \frac{x}{n+x}\right)^{mn}$$
$$\geq \alpha_k(mx) \exp(-k(k-1)/(mn-k+1)),$$

and (i) is proved. To prove (ii) let $u_k = p_k(mn, x) - \alpha_k(mx)$, and $\xi_k = \alpha_k(mx)(\exp(mx^2/(n+x)) - 1)$, $\eta_k = \alpha_k(mx)(1 - \exp(-k(k-1)/(mn - k+1)))$. Since $-\eta_k \leq u_k \leq \xi_k$ from (i), and $|u_k| \leq \xi_k + \eta_k$, we have

$$\sum_{k=0}^{mn} |u_k| \leq \sum_{k=0}^{mn} \xi_k + \sum_{k=0}^{mn} \eta_k. \tag{11}$$

It is obvious that

$$\sum_{k=0}^{mn} \xi_k \leqslant (\exp(mx^2/(n+x))-1) \to 0 \qquad \text{as} \quad n \to \infty. \tag{12}$$

Since $\exp(-v) \geqslant 1-\min(1, v)$ $(v \geqslant 0)$, it follows that

$$\sum_{k=0}^{mn} \eta_k \leqslant \sum_{k=0}^{mn} \alpha_k(mx) \min\left(1, \frac{k(k-1)}{mn-k+1}\right)$$
$$\leqslant \sum_{k=2}^{[\sqrt{mn+1}]} \alpha_k(mx) \frac{k(k-1)}{mn-k+1} + P(W > \sqrt{mn+1}),$$

where W is a Poisson random variable with mean mx. It is easy to verify that

$$\sum_{k=0}^{mn} \eta_k \leqslant \frac{(mx)^2}{(mn+1-\sqrt{mn+1})} + P(W > \sqrt{mn+1}) \to 0 \qquad \text{as} \quad n \to \infty. \tag{13}$$

Hence (ii) follows from (11), (12), and (13), and (ii) implies (iii).

THEOREM 3. *Let* L_n *and* S_m *be respectively defined by* (1) *and* (10) *for* $f \in C_B[0, \infty)$. *Then for each* $x \in [0, \infty)$ *and for a fixed integer* m,

$$L_{mn}\left(f\left(\frac{nx}{1+x}\right), \frac{x}{n}\right) \to S_m(f, x) \qquad \textit{as} \quad n \to \infty. \tag{14}$$

Proof. From (1) and (10) we have

$$L_{mn}\left(f\left(\frac{nx}{1+x}\right), \frac{x}{n}\right) = \sum_{k=0}^{mn} f\left(\frac{nk}{mn+1}\right) p_k(mn, x) = \sigma_n(m, x),$$

and

$$S_m(f, x) = \sum_{k=0}^{mn} f\left(\frac{k}{m}\right) \alpha_k(mx) + \sum_{k=mn+1}^{\infty} f\left(\frac{k}{m}\right) \alpha_k(mx)$$
$$= Q_n(m, x) + R_n(m, x).$$

Thus

$$\left| L_{nm}\left(f\left(\frac{nx}{1+x}\right), \frac{x}{n}\right) - S_m(f, x)\right| \leqslant |\sigma_n(m, x) - Q_n(m, x)| + |R_n(m, x)|. \tag{15}$$

Since f is bounded, we have

$$|R_n(m, x)| \leqslant M \sum_{k=mn+1}^{\infty} \alpha_k(mx) \to 0 \quad \text{as} \quad n \to \infty. \tag{16}$$

Moreover,

$$\begin{aligned}
|\sigma_n(m, x) - Q_n(m, x)| &= \left| \sum_{k=0}^{mn} f\left(\frac{nk}{mn+1}\right) p_k(mn, x) \right. \\
&\quad \left. - \sum_{k=0}^{mn} f\left(\frac{k}{m}\right) \alpha_k(mx) \right| \\
&\leqslant \sum_{k=0}^{mn} \left| f\left(\frac{nk}{mn+1}\right) \right| |p_k(mn, x) - \alpha_k(mx)| \\
&\quad + \sum_{k=0}^{mn} \left| f\left(\frac{nk}{mn+1}\right) - f\left(\frac{k}{m}\right) \right| \alpha_k(mx) \\
&\leqslant M \sum_{k=0}^{mn} |p_k(mn, x) - \alpha_k(mx)| \\
&\quad + \sum_{k=0}^{mn} \left| f\left(\frac{nk}{mn+1}\right) - f\left(\frac{k}{m}\right) \right| \alpha_k(mx).
\end{aligned}$$

Since f is uniformly continuous on $[0, \infty)$, it is easy to show that given $\varepsilon > 0$ there exists an integer N_0 such that for $n \geqslant N_0$,

$$\sum_{k=0}^{mn} \left| f\left(\frac{nk}{mn+1}\right) - f\left(\frac{k}{m}\right) \right| \alpha_k(mx) \leqslant \varepsilon.$$

Thus for $n \geqslant N_0$ we have

$$|\sigma_n(m, x) - Q_n(m, x)| \leqslant M \sum_{k=0}^{mn} |p_k(mn, x) - \alpha_k(mx)| + \varepsilon.$$

Hence by the preceding lemma we obtain

$$|\sigma_n(m, x) - Q_n(m, x)| \to 0 \quad \text{as} \quad n \to \infty, \tag{17}$$

and (14) follows from (15), (16), and (17).

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